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A HASSE PRINCIPLE FOR FUNCTION FIELDS OVER PAC FIELDS

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ABSTRACT

Let K be a perfect pseudo-algebraically closed field and let F be an extension of K of relative transcendence degree 1. It is shown that the restriction map $\mathrm{Res}\colon \mathrm{Br}(F) \to \prod_{\mathfrak{p}} \mathrm{Br}(F_{\mathfrak{p}}^h)$ is injective, where $\mathfrak{p}$ ranges over all non-trivial K-places of F, and $F_{\mathfrak{p}}^h$ is the corresponding henselization. Conversely, the validity of this Hasse principle for all such extensions F implies a weaker version of pseudo-algebraic closedness. As an application we determine the finitely generated pro-p closed subgroups of the absolute Galois group of $K(t)$.

Introduction

A field K is called **pseudo-algebraically closed** (PAC) if every geometrically irreducible affine variety defined over K has a K-rational point. For example, the following fields are known to be PAC:

(i) Infinite extensions of finite fields (Eršov [FJ, Cor. 10.5]).

(ii) The fixed field $\tilde{\mathbb{Q}}(\sigma_1,\dots,\sigma_n)$ in the algebraic closure $\tilde{\mathbb{Q}}$ of $\mathbb{Q}$ of almost all automorphisms $\sigma_1,\dots,\sigma_n$ of $\tilde{\mathbb{Q}}$; here "almost all" is in the sense of the canonical Haar measure on the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ (Jarden [FJ, Th. 16.18]).

(iii) $\mathbb{Q}_{\mathrm{tr}}(\sqrt{-1})$, where $\mathbb{Q}_{\mathrm{tr}}$ is the field of all totally real algebraic numbers (Pop [P2]).

(iv) The infinite models of the first-order theory of finite fields (Ax [Ax]).

(v) Algebraic extensions of PAC fields (Ax–Roquette [FJ, Cor. 10.7]).

If a perfect field is PAC then it is infinite, non-real, and all its henselizations with respect to non-trivial valuations are algebraically closed ([FJ, §10.5], [F]).

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It is a long-standing open problem whether the converse holds [FJ, Problem 10.6(b)]. Thus the PAC fields can be considered as the opposite extreme to local fields, in the sense that they do not carry any "essential" arithmetic objects.

In this paper we study field-extensions F of relative transcendence degree 1 over a perfect PAC field K. We prove that the **Hasse principle** for the Brauer groups holds for the extension F/K; i.e., the restriction map

$$\mathrm{Res}\colon \mathrm{Br}(F) \to \prod_{\mathfrak{p}} \mathrm{Br}(F_{\mathfrak{p}}^h)$$

is injective, where $\mathfrak{p}$ ranges over all non-trivial K-places of F and $F_{\mathfrak{p}}^h$ is the corresponding henselization. This complements well-known Hasse principles for Brauer groups of transcendence degree 1 extensions of local fields due to Witt [Wt], Tate [L, Ch. X, Prop. 6.4], Lichtenbaum [Li], Saito [Sa], and Pop [P1, §4].

As a partial converse we show that if the Hasse principle for the Brauer groups holds for all transcendence degree 1 extensions F of a perfect field K, then all the henselizations of K with respect to non-trivial valuations having residue characteristic 0 are algebraically closed.

As an application we characterize the finitely generated pro-p (closed) subgroups of $G_{K(t)}$ for K, perfect and PAC. This gives a new evidence, in an important test-case, for the "pro-p elementary type conjecture" introduced in [E1], which predicts an internal group-theoretic description of the finitely generated maximal pro-p Galois groups of fields (see also [E4]–[E6] for more details and evidence).

We refer to [FJ] for many more examples and applications of PAC fields in Galois theory and in the model theory of fields.

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Note (September 1999): Using among other things the results of this paper, W.-D. Geyer and M. Jarden have meanwhile answered [FJ, Problem 10.6(b)] negatively. Their construction also answers negatively Question 4.2 below.

Conventions

We denote the algebraic, separable, and inseparable closures of a field K by $\tilde{K}$, K_{sep}, and K_{ins}, respectively. Let $G_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ be the absolute Galois group of K. Given a prime number p, let $K(p)$ be the composite of all finite Galois extensions of K of p-power order, and let $G_K(p) = \mathrm{Gal}(K(p)/K)$ be the maximal pro-p Galois group of K. For a (Krull) valuation v on K let $\bar{K}_v$, Γ_v, O_v, and $\mathfrak{m}_v$ be the corresponding residue field, value group, valuation ring, and valuation ideal, respectively. We do not distinguish between equivalent valuations.

Varieties over a field K are always understood as projective, smooth, and geometrically irreducible (with the obvious exception of the affine varieties mentioned in the definition of a PAC field). The set of K-rational points of a variety V is denoted by $V(K)$.

All cohomology groups are in the profinite sense, as in [Se]. Given a field extension E/F we write $\mathrm{Br}(E/F)$ for the kernel of the restriction map $\mathrm{Res}\colon \mathrm{Br}(F) \to \mathrm{Br}(E)$. We denote the Pontrjagin dual $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ of a profinite group G by $G^\vee$. We denote the free pro-p product of pro-p groups $G_1, \dots, G_n$ by $G_1 *_p \cdots *_p G_n$. For an abelian group A and a positive integer n let ${}_nA$ and A/n be the kernel and cokernel, respectively, of the map $A \to A$ of multiplication by n. For a prime number p let $A_p = \bigcup_{i=1}^{\infty} {}_{p^i}A$ be the p-primary component of A.

1. Cohomology of function fields

Fix a perfect field K and a regular extension F of K of relative transcendence degree 1. Let $\mathbb{P}(F/K)$ be the set of all non-trivial valuations v on F which are trivial on K. Thus $\mathbb{P}(F/K)$ may be identified with the set of all K-places $\pi\colon F \to \tilde{K} \cup \{\infty\}$. For each $v \in \mathbb{P}(F/K)$ we fix a henselization (F_v^h, v^h) of (F, v). It is an immediate extension of (F, v) (i.e., has the same residue field and value group as (F, v)). Its inertia field in F_{sep} is $\tilde{K}F_v^h$. One has $\bar{F}_v = \tilde{K} \cap F_v^h$. Hence $\mathrm{Gal}(\tilde{K}F_v^h/F_v^h) \cong G_{\bar{F}_v}$ via restriction. The inclusion $G_{\bar{F}_v} \hookrightarrow G_K$ is compatible with the inclusion map $(\tilde{K}F)^\times \hookrightarrow (\tilde{K}F_v^h)^\times$ and with the valuation map $(\tilde{K}F)^\times/\tilde{K}^\times \to \Gamma_v$ in the sense of [Se, I-11]. By Tsen's theorem we therefore have:

LEMMA 1.1: *One has a natural commutative diagram:*

$$\begin{array}{ccc}
\mathrm{Br}(F) & \xrightarrow{\mathrm{Res}} & \prod_{v\in\mathbb{P}(F/K)} \mathrm{Br}(F_v^h) \\
\cong\big\downarrow & & \big\downarrow\cong \\
H^2(G_K, (\tilde{K}F)^\times) & \longrightarrow & \prod_{v\in\mathbb{P}(F/K)} H^2(G_{\bar{F}_v}, (\tilde{K}F_v^h)^\times) \\
\big\downarrow & & \big\downarrow \\
H^2(G_K, (\tilde{K}F)^\times/\tilde{K}^\times) & \xrightarrow{\mathrm{val}_F} & \prod_{v\in\mathbb{P}(F/K)} H^2(G_{\bar{F}_v}, \Gamma_v)
\end{array}$$

Remark 1.2: Let $v \in \mathbb{P}(F/K)$. Since Γ_v is an ordered group, its divisible hull $\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$ is uniquely divisible, hence cohomologically trivial. Therefore

$$H^2(G_{\bar{F}_v}, \Gamma_v) \cong H^1(G_{\bar{F}_v}, (\Gamma_v \otimes \mathbb{Q})/\Gamma_v) = \mathrm{Hom}(G_{\bar{F}_v}, (\Gamma_v \otimes \mathbb{Q})/\Gamma_v).$$

When $\Gamma_v \cong \mathbb{Z}$ one thus has $H^2(G_{\bar{F}_v}, \Gamma_v) \cong G_{\bar{F}_v}^{\vee}$.

We now assume further that F is a function field in one variable over the perfect field K; i.e., F/K is a regular finitely generated extension of relative transcendence degree 1. Then all valuations in $\mathbb{P}(F/K)$ are discrete. One defines

$$\mathrm{Div}(F/K) = \bigoplus_{v \in \mathbb{P}(F/K)} \Gamma_v \cong \bigoplus_{v \in \mathbb{P}(F/K)} \mathbb{Z}.$$

Recall that for the homomorphism $\mathrm{div}\colon F^\times \to \mathrm{Div}(F/K)$, $f \mapsto (v(f))_v$, we have $K^\times = \mathrm{Ker}(\mathrm{div})$ [H, I, Th. 3.4(a)]. As usual, we denote $\mathrm{Pic}(F/K) = \mathrm{Coker}(\mathrm{div})$. The degree homomorphism $\deg\colon \mathrm{Div}(F/K) \to \mathbb{Z}$ is defined by $\deg((n_v)_v) = \sum_v n_v[\bar{F}_v : K]$, for $n_v \in \mathbb{Z}$. It induces a homomorphism $\deg\colon \mathrm{Pic}(F/K) \to \mathbb{Z}$ whose kernel is denoted by $\mathrm{Pic}^0(F/K)$.

Next we recall the G_K-module structure of $\mathrm{Div}(\tilde{K}F/\tilde{K})$. Any $v \in \mathbb{P}(F/K)$ has a (usually non-unique) prolongation w to $\mathbb{P}(\tilde{K}F/\tilde{K})$ and $\Gamma_w = \Gamma_v$. The group G_K acts transitively from the right on the set of prolongations w of v by $w^\sigma = w \circ \sigma$. The G_K-stabilizer of any such prolongation w is the restriction to $\tilde{K}$ of the decomposition group $Z(w/v)$ of w in $\mathrm{Gal}(\tilde{K}F/F)$. When w is the restriction w_v to $\tilde{K}F$ of the unique prolongation of v^h to $\tilde{K}F_v^h$, this restriction is $G_{\bar{F}_v}$.

Now let $\langle v \rangle = \bigoplus_w \Gamma_w = \bigoplus_w \Gamma_v$, where w ranges over all prolongations of v to $\tilde{K}F$. We view $\langle v \rangle$ as a G_K-module in the natural way. By what we have just seen, $\langle v \rangle$ may be identified with the group of all maps $h\colon G_K \to \Gamma_v$ such that $h(\sigma\tau) = h(\tau)$ for $\sigma \in G_{\bar{F}_v}$, $\tau \in G_K$ (with the topology on Γ_v being discrete); namely, $(\gamma_w)_w \in \bigoplus_w \Gamma_v$ corresponds to the map h given by $h(\tau) = \gamma_{w_v \circ \tau}$. The action of G_K on $\langle v \rangle$ is then given by $(\rho h)(\tau) = h(\tau\rho)$ for $\rho, \tau \in G_K$. Thus $\langle v \rangle$ is the induced module $\mathrm{Ind}_{G_K}^{G_{\bar{F}_v}}(\Gamma_v)$, with Γ_v considered as a trivial $G_{\bar{F}_v}$-module [Se, I, §2.5]. We obtain a G_K-module isomorphism

$$\mathrm{Div}(\tilde{K}F/\tilde{K}) = \bigoplus_{v \in \mathbb{P}(F/K)} \langle v \rangle \cong \bigoplus_{v \in \mathbb{P}(F/K)} \mathrm{Ind}_{G_K}^{G_{\bar{F}_v}}(\Gamma_v).$$

Define $\lambda_v\colon \mathrm{Ind}_{G_K}^{G_{\bar{F}_v}}(\Gamma_v) \to \Gamma_v$ by $\lambda_v(h) = h(1)$. By Shapiro's lemma [Se, I-12, Prop. 10], it induces isomorphisms $\lambda_v^l\colon H^l(G_K, \mathrm{Ind}_{G_K}^{G_{\bar{F}_v}}(\Gamma_v)) \xrightarrow{\sim} H^l(G_{\bar{F}_v}, \Gamma_v)$. For $l = 0$ this gives $\mathrm{Div}(\tilde{K}F/\tilde{K})^{G_K} = \mathrm{Div}(F/K)$. Consequently, one may identify $\mathrm{Pic}(F/K)$ as a subgroup of $\mathrm{Pic}(\tilde{K}F/\tilde{K})^{G_K}$.

PROPOSITION 1.3: *Let F be a function field in one variable over a perfect field K. One has a natural exact sequence:*

$$0 \to H^1(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K})) \to H^2(G_K, (\tilde{K}F)^\times/\tilde{K}^\times) \longrightarrow \bigoplus_{v \in \mathbb{P}(F/K)} H^2(G_{\bar{F}_v}, \Gamma_v)$$

$$\to H^2(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K})) \to H^3(G_K, (\tilde{K}F)^\times/\tilde{K}^\times).$$

Proof: For every $v \in \mathbb{P}(F/K)$, $G_{\bar{F}_v}$ is compact and Γ_v is discrete and torsion-free. Hence

$$H^1(G_K, \mathrm{Ind}_{G_K}^{G_{\bar{F}_v}}(\Gamma_v)) \cong H^1(G_{\bar{F}_v}, \Gamma_v) = \mathrm{Hom}(G_{\bar{F}_v}, \Gamma_v) = 0.$$

The assertion therefore follows from the cohomology exact sequence corresponding to the short exact sequence of G_K-modules:

$$1 \to (\tilde{K}F)^\times/\tilde{K}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\tilde{K}F/\tilde{K}) = \bigoplus_{v \in \mathbb{P}(F/K)} \mathrm{Ind}_{G_K}^{G_{\bar{F}_v}}(\Gamma_v) \to \mathrm{Pic}(\tilde{K}F/\tilde{K}) \to 0. \qquad \blacksquare$$

LEMMA 1.4: *Let F be a function field in one variable over a perfect field K and let p be a prime number.*

(a) *The natural homomorphism*

$$H^1(G_K, \mathrm{Pic}^0(\tilde{K}F/\tilde{K})) \to H^1(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K}))$$

is surjective.

(b) $H^i(G_K, \mathrm{Pic}^0(\tilde{K}F/\tilde{K}))_p = 0$ *for* $i > \mathrm{cd}_p(G_K)$.

(c) $H^i(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K}))_p \cong H^{i-1}(G_K, \mathbb{Q}/\mathbb{Z})_p$ *naturally for* $i > \max\{\mathrm{cd}_p(G_K), 1\}$.

Proof: We use the short exact sequence of G_K-modules:

$$(*) \qquad 0 \to \mathrm{Pic}^0(\tilde{K}F/\tilde{K}) \longrightarrow \mathrm{Pic}(\tilde{K}F/\tilde{K}) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

(a) As G_K is compact and the group $\mathbb{Z}$ is discrete and torsion-free, one has $H^1(G_K, \mathbb{Z}) = \mathrm{Hom}(G_K, \mathbb{Z}) = 0$. Now use the cohomology exact sequence corresponding to $(*)$.

(b) Let $J = \mathrm{Jac}_{F/K}$ be the Jacobian variety of F/K. The group $J(\tilde{K})$ is divisible [Mu, p. 62]. For every $n \in \mathbb{N}$ consider the cohomology exact sequence corresponding to

$$0 \to {}_{p^n}J(\tilde{K}) \to J(\tilde{K}) \xrightarrow{p^n} J(\tilde{K}) \to 0.$$

It shows that for every $i > \mathrm{cd}_p(G_K)$, the group $H^i(G_K, J(\tilde{K}))_p$ is uniquely divisible. Since it is also a torsion group, it is trivial. Finally, $J(\tilde{K}) \cong \mathrm{Pic}^0(\tilde{K}F/\tilde{K})$ as G_K-modules [Li, Lemma 1].

(c) By $(*)$ again and (b), deg induces isomorphisms $H^i(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K}))_p \cong H^i(G_K, \mathbb{Z})_p$ for $i > \mathrm{cd}_p(G_K)$. As $\mathbb{Q}$ is uniquely divisible, $H^l(G_K, \mathbb{Q}) = 0$ for all $l \geq 1$. Hence $H^{i-1}(G_K, \mathbb{Q}/\mathbb{Z}) \cong H^i(G_K, \mathbb{Z})$ when $i > \max\{\mathrm{cd}_p(G_K), 1\}$. ∎

We record the following fact which is proven in [Sc, §1] (modulo the q-primary components if $q = \mathrm{char}\,\bar{E}_v > 0$) and in [P1, Th. 2.4] (in general):

PROPOSITION 1.5: *Let (E, v) be a henselian valued field and let E_t be its inertia field in E_{sep}. The valuation map $E_t^\times \to \Gamma_v$ induces an exact sequence*

$$0 \to \mathrm{Br}(\bar{E}_v) \to H^2(G_{\bar{E}_v}, E_t^\times) \to H^2(G_{\bar{E}_v}, \Gamma_v) \to 0.$$

2. Fields of cohomological dimension ≤ 1

If K is PAC then $\mathrm{cd}(G_K) \leq 1$ [FJ, Th. 10.17]. We analyze the exact sequence of Proposition 1.3 for fields satisfying the latter weaker condition. We recall that the strict cohomological dimension $\mathrm{scd}(G)$ of a profinite group G is the minimal non-negative integer n (∞ if no such integer exists) such that $H^q(G, A) = 0$ for every $q > n$ and every discrete G-module A [Se, I-18].

LEMMA 2.1: *Let K be a perfect field with $\mathrm{cd}(G_K) \leq 1$ and let F be a regular extension of K with $\mathrm{tr.deg}(F/K) = 1$. Then:*

(a) *$H^2(G_K, (\tilde{K}F)^\times) \cong H^2(G_K, (\tilde{K}F)^\times/\tilde{K}^\times)$ naturally;*
(b) *$H^i(G_K, (\tilde{K}F)^\times/\tilde{K}^\times) = 0$ for $i \geq 3$;*
(c) *for every $v \in \mathbb{P}(F/K)$ the valuation map $H^2(G_{\bar{F}_v}, (\tilde{K}F_v^h)^\times) \to H^2(G_{\bar{F}_v}, \Gamma_v)$ is an isomorphism.*

Proof: Since $\mathrm{cd}(G_K) \leq 1$ and K is perfect, $H^2(G_K, \tilde{K}^\times) = \mathrm{Br}(K) = 0$ [Se, II-8, Prop. 6(b)]. Further, $\mathrm{scd}(G_K) \leq 2$ [Se, I-19, Prop. 13], so $H^i(G_K, \tilde{K}^\times) = H^i(G_K, (\tilde{K}F)^\times) = 0$ for $i \geq 3$. (a) and (b) therefore follow from the cohomology sequence corresponding to the short exact sequence

$$1 \to \tilde{K}^\times \to (\tilde{K}F)^\times \to (\tilde{K}F)^\times/\tilde{K}^\times \to 1$$

of discrete G_K-modules.

Finally take $v \in \mathbb{P}(F/K)$. Since $\bar{F}_v$ is an algebraic extension of K, it is perfect and $\mathrm{cd}(G_{\bar{F}_v}) \leq 1$, so again $\mathrm{Br}(\bar{F}_v) = 0$. Therefore (c) follows from Proposition 1.5. ∎

COROLLARY 2.2: *Let K be a perfect field with $\mathrm{cd}(G_K) \leq 1$ and let F be a regular extension of K with $\mathrm{tr.deg}(F/K) = 1$. There is a natural commutative square with bijective vertical maps:*

$$\begin{array}{ccc} \mathrm{Br}(F) & \xrightarrow{\mathrm{Res}} & \prod_{v\in\mathbb{P}(F/K)} \mathrm{Br}(F_v^h) \\ \cong\downarrow & & \downarrow\cong \\ H^2(G_K, (\tilde{K}F)^\times/\tilde{K}^\times) & \longrightarrow & \prod_{v\in\mathbb{P}(F/K)} H^2(G_{\tilde{F}_v}, \Gamma_v) \end{array}$$

Proof: Apply Lemma 1.1 and Lemma 2.1(a)(c). ∎

PROPOSITION 2.3: *Let K be a perfect field with $\mathrm{cd}(G_K) \leq 1$ and let F be a function field in one variable over K. One has a natural exact sequence:*

$$0 \to H^1(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K})) \longrightarrow \mathrm{Br}(F) \xrightarrow{\mathrm{Res}} \bigoplus_{v\in\mathbb{P}(F/K)} \mathrm{Br}(F_v^h) \to G_K^\vee \to 0.$$

Proof: By Lemma 1.4(c), $H^2(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K})) \cong G_K^\vee$. Also, Lemma 2.1(b) shows that $H^3(G_K, (\tilde{K}F)^\times/\tilde{K}^\times) = 0$. Now apply Corollary 2.2 and Proposition 1.3. ∎

2.4 Remarks: (a) In the special case where $G_K \cong \hat{\mathbb{Z}}$, most of Proposition 2.3 is proven in [RW, §1].

(b) Still in the case $G_K \cong \hat{\mathbb{Z}}$ and for $m \in \mathbb{N}$ relatively prime to char K, Wiesend [W2, §6.3] obtains for F as above an exact sequence

$$0 \to \pi^{\mathrm{cs}}(F)/m \to {}_m\mathrm{Br}(F) \to \bigoplus_{v\in\mathbb{P}(F/K)} {}_m\mathrm{Br}(F_v^h) \to \mathbb{Z}/m \to 0.$$

Here $\pi^{\mathrm{cs}}(F) = \varprojlim \mathrm{Gal}(E/F)$, where E ranges over all finite abelian Galois extensions of F in which every $v \in \mathbb{P}(F/K)$ splits completely. Rim and Whaples show that any function field in one variable F over $K = \mathbb{C}((t))$ with positive genus has such a proper extension E [RW, §2]. Since $G_{\mathbb{C}((t))} \cong \hat{\mathbb{Z}}$, this shows that in the exact sequence of Proposition 2.3 the term $H^1(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K}))$ can be non-trivial. See also [D] for related results.

3. The Hasse principle

Let A be an abelian variety defined over a field K. Recall that a **principal homogeneous space** for A over K is a variety V over K with a simply transitive right action of A on V such that the map $\nu\colon V \times V \to A$, which takes (x, x') to the unique element $a \in A$ for which $x = x'a$, is rational over K. An isomorphism of principal homogeneous spaces V and V' for A over K is an isomorphism of varieties $f\colon V \to V'$ which is defined over K and such that $f(xa) = f(x)a$ for all $x \in V$ and $a \in A$.

The set of isomorphism classes of principal homogeneous spaces for A over K can be canonically identified with $H^1(G_K, A(K_{\mathrm{sep}}))$. This bijection is given as follows: a principal homogeneous space V corresponds to the cohomology class of the 1-cocycle $\sigma \mapsto \nu(p_0^\sigma, p_0)$ in $Z^1(G_K, A(K_{\mathrm{sep}}))$, where p_0 is any fixed K_{sep}-rational point on V and ν is as above [LT, Prop. 4]. The class of A, viewed as a principal homogeneous space for itself in the natural way, then corresponds to the zero element of $H^1(G_K, A(K_{\mathrm{sep}}))$ (since we can take $p_0 = 0 \in A$). More generally, the isomorphism class of a principal homogeneous space V corresponds to the zero element of $H^1(G_K, A(K_{\mathrm{sep}}))$ if and only if $V(K) \neq \emptyset$. Indeed, if $f\colon A \to V$ is an isomorphism of principal homogeneous spaces for A over K then $f(0) \in V(K)$; conversely, if $p_0 \in V(K)$ then the 1-cocycle above is trivial.

PROPOSITION 3.1: *Let A be an abelian variety defined over a PAC field K. Then $H^1(G_K, A(K_{\mathrm{sep}})) = 0$.*

Proof: Since K is PAC every (projective) variety defined over it has K-rational points [F]. In particular, every principal homogeneous space for A over K has K-rational points, hence it corresponds to the zero element of $H^1(G_K, A(K_{\mathrm{sep}}))$. ∎

We now obtain our Hasse principle for the Brauer groups in the special case of function fields:

COROLLARY 3.2: *Let F be a function field in one variable over a perfect PAC field K. There is a natural exact sequence*

$$0 \to \mathrm{Br}(F) \xrightarrow{\mathrm{Res}} \bigoplus_{v \in \mathbb{P}(F/K)} \mathrm{Br}(F_v^h) \to G_K^\vee \to 0.$$

Proof: By Proposition 3.1, $H^1(G_K, \mathrm{Jac}_{F/K}(\tilde{K})) = 0$. The isomorphism

$$\mathrm{Jac}_{F/K}(\tilde{K}) \cong \mathrm{Pic}^0(\tilde{K}F/\tilde{K})$$

of G_K-modules and Lemma 1.4(a) show that $H^1(G_K, \mathrm{Pic}(\tilde{K}F/\tilde{K})) = 0$. Now apply Proposition 2.3. ∎

In order to generalize this Hasse principle to arbitrary extensions of relative transcendence degree 1 we need the following general fact:

LEMMA 3.3: *Let P be a set of valuations on a field F and let F_i, $i \in I$, be a system of subfields of F which is directed by inclusion and such that $F = \bigcup_{i\in I} F_i$. For every $i \in I$ let P_i be the set of restrictions of the valuations in P to F_i. Suppose p is a prime number such that for each $i \in I$*

$$\mathrm{Res}\colon \mathrm{Br}(F_i)_p \to \prod_{w\in P_i} \mathrm{Br}((F_i)^h_w)$$

is injective with image contained in the direct sum $\bigoplus_{w\in P_i} \mathrm{Br}((F_i)^h_w)$. Then

$$\mathrm{Res}\colon \mathrm{Br}(F)_p \to \prod_{v\in P} \mathrm{Br}(F^h_v)$$

is injective as well.

Proof: Take $x \in \mathrm{Br}(F)_p$ such that $\mathrm{Res}_{F^h_v}(x) = 0$ for all $v \in P$. Since $\mathrm{Br}(F)_p = \varinjlim \mathrm{Br}(F_i)_p$ there exist $i_0 \in I$ and $x_0 \in \mathrm{Br}(F_{i_0})_p$ such that $x = \mathrm{Res}_F(x_0)$. By assumption there are only finitely many $u \in P_{i_0}$ such that $\mathrm{Res}_{(F_{i_0})^h_u}(x_0) \neq 0$. List them as $u_1, \ldots, u_n$. For each $1 \le j \le n$ a compactness argument yields $i_0 \le i(j) \in I$ such that $\mathrm{Res}_{(F_{i(j)})^h_w}(x_0) = 0$ in $\mathrm{Br}((F_{i(j)})^h_w)$ for every $w \in P_{i(j)}$ extending u_j ([P1, Lemma 4.4], [W1, §4]). Pick $i \in I$ which is larger than $i(1), \ldots, i(n)$. For every $u \in P_i$ we then have $\mathrm{Res}_{(F_i)^h_u}(x_0) = 0$ in $\mathrm{Br}((F_i)^h_u)$. By assumption, this implies $\mathrm{Res}_{F_i}(x_0) = 0$. Hence $x = \mathrm{Res}_F(x_0) = 0$, as desired. ∎

Remark: The assumption in Lemma 3.3 that the images of the restriction maps on the $\mathrm{Br}(F_i)$, $i \in I$, are contained in the direct sums cannot be omitted. E.g., Pop [P1, §4] shows that the Hasse principle for the Brauer groups holds for all function fields in one variable over $\mathbb{Q}_l$, but fails for certain extensions of $\mathbb{Q}_l$ of transcendence degree 1; see also Theorem 4.1 below.

We now obtain our main result:

THEOREM 3.4: *Let F be an extension of a perfect PAC field K of relative transcendence degree 1. Then the Hasse principle for the Brauer groups holds for F/K.*

Proof: Since $\tilde{K} \cap F$ is also a perfect PAC field [FJ, Cor. 10.7], we may assume without loss of generality that F/K is a regular extension. Now apply Corollary 3.2 and Lemma 3.3 with respect to the collection F_i, $i \in I$, of all function fields in one variable contained in F. ∎

3.5 Remarks: (a) Suppose that K is a *non-perfect* PAC field of characteristic $q > 0$ and F is an extension of K of relative transcendence degree 1. Then K_{ins} is perfect and PAC [FJ, Cor. 10.7]. For every prime number $l \neq q$ one has a natural isomorphism $\mathrm{Br}(F)_l \cong H^2(G_F, \mu_{l^\infty})$, where μ_{l^∞} is the G_F-module of all roots of unity of l-power order (over the prime field of F), and similarly for $\mathrm{Br}(F_{\mathrm{ins}})$. Since $G_{F_{\mathrm{ins}}} \cong G_F$ via restriction, $\mathrm{Br}(F_{\mathrm{ins}}/F)$ is q-primary. Further, every $v \in \mathbb{P}(F/K)$ has a unique prolongation to a valuation v_{ins} in $\mathbb{P}(F_{\mathrm{ins}}/K)$ [B, VI, §8.6, Cor. 2]. By applying Theorem 3.4 with respect to F_{ins}, we conclude that the Hasse principle for F/K holds except possibly on the q-primary component.

(b) Away from the q-primary component if $q = \mathrm{char}\, K > 0$, one can replace in the Hasse principle of Theorem 3.4 the henselization (F_v^h, v^h) by the completion (F_v^*, v^*) of (F, v). Indeed, by Krasner's lemma, $(F_v^*)_{\mathrm{sep}} = F_{\mathrm{sep}} F_v^*$, so $\mathrm{Res}\colon G_{F_v^*} \to G_{F_{\mathrm{sep}} \cap F_v^*}$ is an isomorphism. As in (a) this implies that $\mathrm{Res}\colon \mathrm{Br}(F_{\mathrm{sep}} \cap F_v^*) \to \mathrm{Br}(F_v^*)$ is an isomorphism except possibly on the q-primary components. For a suitable embedding, $F_v^h \subseteq F_v^*$; then $(F_v^*, v^*)/(F_v^h, v^h)$ is an immediate extension. It follows from the fundamental equality of valuation theory that $[F_{\mathrm{sep}} \cap F_v^* : F_v^h] | q^\infty$ if $q > 0$, and $F_{\mathrm{sep}} \cap F_v^* = F_v^h$ if $\mathrm{char}\, K = 0$ [Ed, Th. 20.21]. By [Se, I-11, Cor.] the kernel of $\mathrm{Res}\colon \mathrm{Br}(F_v^h) \to \mathrm{Br}(F_{\mathrm{sep}} \cap F_v^*)$ is therefore q-primary if $\mathrm{char}\, K > 0$, and trivial if $\mathrm{char}\, K = 0$. Conclude that so is the kernel of $\mathrm{Res}\colon \mathrm{Br}(F_v^h) \to \mathrm{Br}(F_v^*)$.

(c) It is clear from the proof that Theorem 3.4 remains true if one only assumes that K is a perfect field with $\mathrm{cd}(G_K) \leq 1$ and all principal homogeneous spaces for abelian varieties over K have K-rational points.

(d) The perfect PAC fields with $G_K \cong \hat{\mathbb{Z}}$ are precisely the infinite models of the first-order theory of finite fields [Ax]. When K is a finite field, the Hasse principle for transcendence degree 1 extensions of K holds by global class field theory. Conclude that this Hasse principle for the Brauer groups holds whenever K is a model of the theory of finite fields.

For a field F we abbreviate $H^l(F) = H^l(G_F(p), \mathbb{Z}/p)$.

COROLLARY 3.6: *Let K be a PAC field and let F be an extension of K of relative transcendence degree 1. Then the restriction homomorphism*

$$\mathrm{Res}\colon H^2(F) \to \prod_{v \in \mathbb{P}(F/K)} H^2(\hat{F}_v)$$

is injective, where $\hat{F}_v = F(p) \cap F_v^h$.

Proof: If $\operatorname{char} F = p$ then $H^2(F) = 0$ [Se, II-4, Prop. 3], and we are done. So suppose $\operatorname{char} F \neq p$. If F contains the group μ_p of pth roots of unity then $H^2(F) \cong {}_p\mathrm{Br}(F)$ naturally, and likewise for $\hat{F}_v$ [JWd, (1.7)]. The assertion therefore follows in this case from Theorem 3.4.

When $\mu_p \not\subseteq F$ the assertion holds with F replaced by $F(\mu_p)$. By [E4, Lemma 1.1], $\mathrm{Res}\colon H^2(F) \to H^2(F(\mu_p))$ is injective (we note that the assumption there that $\mu_p \subseteq K$ is superfluous). It remains to notice that every henselization of a valuation in $\mathbb{P}(F(\mu_p)/K(\mu_p))$ contains a henselization of a valuation in $\mathbb{P}(F/K)$. ∎

4. A converse result

In this section we generalize an argument of Pop [P1, Beispiel 4.2] to give a partial converse of Theorem 3.4. Given valuations v, w on a field F one says that v is **coarser** than w if $O_w \subseteq O_v$. Then $\mathfrak{m}_w \supseteq \mathfrak{m}_v$ and $O_w/\mathfrak{m}_v$ is a valuation ring on $\bar{F}_v = O_v/\mathfrak{m}_v$. One denotes the corresponding valuation by w/v. The value group $\Gamma_{w/v} = (O_v/\mathfrak{m}_v)^\times/(O_w/\mathfrak{m}_v)^\times$ embeds in $\Gamma_w = F^\times/O_w^\times$ via $(a+\mathfrak{m}_v)(O_w/\mathfrak{m}_v)^\times \mapsto aO_w^\times$ for $a \in O_v^\times$, and $\Gamma_w/\Gamma_{w/v} \cong \Gamma_v$ [B, Ch. VI, §4.3].

THEOREM 4.1: *Let K be a perfect field such that the Hasse principle for the Brauer groups holds for all extensions F of K of relative transcendence degree 1. Then every non-trivial valuation u on K has divisible value group and algebraically closed residue field. In particular, if* $\operatorname{char} \bar{K}_u = 0$ *then (K, u) has algebraically closed henselizations.*

Proof: Let w_0 be the Gauss valuation on $K(t)$; i.e.,

$$w_0(a_0 + a_1 t + \cdots + a_n t^n) = \min\{u(a_0), u(a_1), \ldots, u(a_n)\}$$

for $a_0, a_1, \ldots, a_n \in K$. One has $\Gamma_{w_0} = \Gamma_u$ and $\overline{K(t)}_{w_0} = \bar{K}_u(\bar{t})$, where $\bar{t}$ is the residue of t, which is transcendental over $\bar{K}_u$ [B, Ch. VI, §10.1, Prop. 2]. Fix a henselization (F, w) of $(K(t), w_0)$.

Let v be a valuation on F which is trivial on K and is coarser than w. For every $\gamma \in \Gamma_w$ there exists $a \in K^\times$ with $\gamma = w(a)$. Then $(w/v)(a) = (a+\mathfrak{m}_v)(O_w/\mathfrak{m}_v)^\times$ and this value maps to $\gamma = w(a) = aO_w^\times$ under the natural embedding $\Gamma_{w/v} \hookrightarrow \Gamma_w$. Conclude that this embedding is an isomorphism, whence $\Gamma_v = 0$, i.e., v is trivial.

Consequently, any $v \in \mathbb{P}(F/K)$ is independent of w. Now the henselization (F_v^h, v^h) of (F, v) is also henselian with respect to the unique prolongation of w. Thus it is henselian with respect to two independent non-trivial valuations. By a result of Engler [Eg], F_v^h is necessarily separably closed, whence $\mathrm{Br}(F_v^h)=0$. The Hasse principle implies that $\mathrm{Br}(F)=0$. In particular, $\mathrm{Br}(F_t/F)=0$ for the inertia field F_t of (F,w) relative to F_{sep}.

Proposition 1.5 now implies $\mathrm{Br}(\bar{F}_w)=0$ and $H^2(G_{\bar{F}_w},\Gamma_w)=0$. But the extension $(F,w)/(K(t),w_0)$ is immediate, so the first equality means $\mathrm{Br}(\bar{K}_u(\bar{t}))=0$, which can happen if and only if $\bar{K}_u$ is algebraically closed [AB, Cor. 4.8]. By Remark 1.2, the second equality means that $\mathrm{Hom}(G_{\bar{F}_w},(\Gamma_w\otimes\mathbb{Q})/\Gamma_w)=0$. Since $\bar{F}_w=\bar{K}_u(\bar{t})$ has Galois extensions of order l for every prime number l [FJ, Cor. 14.10 and Cor. 15.9], $\Gamma_w\otimes\mathbb{Q}=\Gamma_w$; i.e., $\Gamma_u=\Gamma_w$ is divisible.

For the last statement of the theorem see [Ed, Th. 20.21]. ∎

QUESTION 4.2: *Let K be a non-real infinite perfect field such that the Hasse principle for the Brauer groups holds for all extensions F of K of relative transcendence degree 1. Is K necessarily PAC?*

5. Finitely generated pro-p Galois groups

From now on we fix a prime number p. We recall from [E5], [E6] (see [JW1], [JW2] for the case $p=2$) the following terminology: a **cyclotomic pro-p pair** (G,θ) consists of a pro-p group G and a continuous homomorphism $\theta\colon G\to 1+p\mathbb{Z}_p$. A **morphism** $\Phi\colon (G,\theta)\to(G',\theta')$ of cyclotomic pro-p pairs is a continuous homomorphism $\Phi\colon G\to G'$ such that $\theta=\theta'\circ\Phi$. We do not distinguish between isomorphic pairs. We say that the pair (G,θ) is **finitely generated** if G is finitely generated as a pro-p group.

Let (G,θ) be a cyclotomic pro-p pair and let m be a cardinal number. The **semi-direct product** $\mathbb{Z}_p^m\rtimes(G,\theta)$ is the cyclotomic pro-p pair $(\mathbb{Z}_p^m\rtimes G,\theta\circ\pi)$, with $\sigma\in G$ acting on $\tau\in\mathbb{Z}_p^m$ by $\tau^\sigma=\theta(\sigma)\tau$, and where π is the natural projection $\mathbb{Z}_p^m\rtimes G\to G$.

The **free product** $\mathcal{G}_1*\cdots*\mathcal{G}_n$ of cyclotomic pro-p pairs $\mathcal{G}_1=(G_1,\theta_1),\ldots,\mathcal{G}_n=(G_n,\theta_n)$ is the pair (G,θ), where $G=G_1*_p\cdots*_pG_n$, and the homomorphism $\theta\colon G\to 1+p\mathbb{Z}_p$ is induced by $\theta_1,\ldots,\theta_n$ via the universal property of G.

Next let K be a field of characteristic $\neq p$ containing μ_p. Let $\mathrm{Aut}(\mu_{p^\infty}/\mu_p)$ be the group of all automorphisms of μ_{p^∞} fixing μ_p. Then $1+p\mathbb{Z}_p\cong\mathrm{Aut}(\mu_{p^\infty}/\mu_p)$, where $\alpha\in 1+p\mathbb{Z}_p$ corresponds to the automorphism $\zeta\mapsto\zeta^\alpha$. The p-cyclotomic character $\chi_{K,p}\colon G_K(p)\to 1+p\mathbb{Z}_p$ is defined as the composition of $\mathrm{Res}\colon G_K(p)\to\mathrm{Aut}(\mu_{p^\infty}/\mu_p)$ with this isomorphism. Then $\mathcal{G}(K)=(G_K(p),\chi_{K,p})$ is a cyclotomic pro-p pair.

5.1 Remarks: Let K be a field of characteristic $\neq p$ containing μ_p.

(a) For subextensions $K \subseteq L_1, \ldots, L_n \subseteq K(p)$ one has

$$\mathcal{G}(K) = \mathcal{G}(L_1) * \cdots * \mathcal{G}(L_n)$$

(via the natural restriction morphisms) if and only if

$$G_K(p) = G_{L_1}(p) *_p \cdots *_p G_{L_n}(p)$$

naturally.

(b) If L is an extension of K and $\mathrm{Res}\colon G_L(p) \to G_K(p)$ is an isomorphism then $\mathcal{G}(L) \cong \mathcal{G}(K)$ via restriction. In particular, if L/K is a purely inseparable algebraic extension then $\mathcal{G}(L) \cong \mathcal{G}(K)$ naturally.

(c) Suppose that v is a valuation on K with $\mathrm{char}\, \bar{K}_v \neq p$ and which is **p-henselian**, in the sense that Hensel's lemma holds for polynomials over K that split completely in $K(p)$. Then $\mathcal{G}(K) \cong \mathbb{Z}_p^{\dim_{\mathbb{F}_p} \Gamma_v/p} \rtimes \mathcal{G}(\bar{K}_v)$ [E3, Lemma 1.1].

(d) We will make repeated use of the following observation: if K is PAC then $\mathrm{cd}(G_K) \leq 1$, so $G_K(p)$ is a free pro-p group [Se1, II-4, Prop. 2 and I-37, Cor. 2]. In particular, $G_K(p)$ is torsion-free and the natural epimorphism $G_K \to G_K(p)$ splits.

For the proof of Theorem 5.6 below we will need the following facts on free pro-p products.

LEMMA 5.2 ([E4, Prop. 1.3]): *The following conditions on closed subgroups $\Gamma_1, \ldots, \Gamma_s$ of a finitely generated pro-p group G are equivalent:*

(a) $G = \Gamma_1 *_p \cdots *_p \Gamma_s *_p H$ *for some free pro-p closed subgroup H of G;*

(b) $\mathrm{Res}\colon H^l(G, \mathbb{Z}/p) \to \prod_{i=1}^s H^l(\Gamma_i, \mathbb{Z}/p)$ *is surjective for $l = 1$ and injective for $l = 2$.*

LEMMA 5.3: *Let K be a field of characteristic $\neq p$ containing μ_p. Let F/K be a regular extension of transcendence degree 1, and let S be a finite subset of $\mathbb{P}(F/K)$. For $v \in S$ let $\hat{F}_v = F(p) \cap F_v^h$. Then $\mathrm{Res}\colon H^1(F) \to \prod_{v \in S} H^1(\hat{F}_v)$ is surjective.*

Proof: (This is implicit in [E3, Prop. 4.3].) In light of the Kummer isomorphisms $H^1(F) \cong F^\times/p$, $H^1(\hat{F}_v) \cong \hat{F}_v^\times/p$, it suffices to show that the natural map $F^\times \to \prod_{v \in S} \hat{F}_v^\times/p$ is surjective. Indeed, take $a_v \in \hat{F}_v^\times$, $v \in S$. Since v^h and v have the same value group and residue field, there exist $b_v \in F^\times$ such that $v^h(a_v) = v(b_v)$ and $b_v/a_v \in 1 + \mathfrak{m}_{v^h}$. Since the valuations in S have rank 1 they are independent, whence there exists $a \in F^\times$ such that $a/b_v \in 1 + \mathfrak{m}_v$ for all $v \in S$ [B, Ch. VI, §7.2, Cor. 1]. As $\operatorname{char} K \neq p$, Hensel's lemma shows that $a/a_v \in F(p) \cap (1 + \mathfrak{m}_{v^h}) \subseteq F(p) \cap (F_v^h)^p = \hat{F}_v^p$ for all $v \in S$. ∎

PROPOSITION 5.4: *Let F be a field of characteristic $\neq p$ which contains μ_p and let $(F_1, v_1), \ldots, (F_n, v_n)$ be p-henselian valued field extensions of F contained in $F(p)$. Suppose that $F = F_1 \cap \cdots \cap F_n$ and that $\operatorname{Res}_F v_1, \ldots, \operatorname{Res}_F v_n$ are non-trivial independent valuations on F. Then $G_F(p) = G_{F_1}(p) *_p \cdots *_p G_{F_n}(p)$.*

Proof: This is a special case of [E3, Prop. 4.3]. ∎

Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p.

LEMMA 5.5: *Let K be a field of characteristic $\neq p$ containing μ_p. There exist henselian valued fields (K_1, v_1), (K_2, v_2) which are algebraic and separable over $K(t)$ such that $\Gamma_{v_1} = \mathbb{Z}_{(p)}$, $\Gamma_{v_2} = \mathbb{Q}$, $(\overline{K_i})_{v_i}/K$ is a purely inseparable algebraic extension, and $v_i(t) > 0$, $i = 1, 2$. Moreover, if G_K is a pro-p group then we can find such K_1, K_2 with G_{K_1}, G_{K_2} pro-p.*

Proof: Let $\Gamma_1 = \mathbb{Z}_{(p)}$ and $\Gamma_2 = \mathbb{Q}$. Given a positive integer n the field $K((t^{1/n}))$ is equipped with a natural K-trivial discrete henselian valuation w_n which is normalized so that $w_n(t^{1/n}) = 1/n$. Let $E_1 = \bigcup K((t^{1/n}))$, with n ranging over all positive integers not divisible by p. Let $E_2 = \bigcup K((t^{1/n}))$, with n ranging over all positive integers. The union of the corresponding valuations w_n is then a valuation u_i' on E_i with residue field K and value group Γ_i, $i = 1, 2$. Being a direct limit of henselian fields, (E_i, u_i') is henselian as well.

Let $F_i = \widetilde{K(t)} \cap E_i$ and let u_i be the restriction of u_i' to F_i. Then (F_i, u_i) as well is henselian with residue field K and value group Γ_i. Let (K_i, v_i) be an algebraic F_i-complement of the ramification field of (F_i, u_i), in the sense of [KPR, §2]. If $\operatorname{char} K = 0$ then $(F_i, u_i) = (K_i, v_i)$. If $q = \operatorname{char} K > 0$ then $q \neq p$ implies $\Gamma_i = q\Gamma_i$. Thus $\Gamma_{v_i} = \Gamma_{u_i} = \Gamma_i$ and $(\overline{K_i})_{v_i}/K$ is a purely inseparable algebraic extension [KPR, Th. 4.3]. Since (K_i, v_i) is henselian with trivial ramification group, its inertia group is $\operatorname{Hom}((\Gamma_i \otimes \mathbb{Q})/\Gamma_i, \tilde{K}^\times)$ [Ed, Th. 20.12]. Thus the inertia group of (K_1, v_1) is $\mathbb{Z}_p$, and that of (K_2, v_2) is trivial. In particular, if G_K is pro-p then so are G_{K_1}, G_{K_2}. ∎

THEOREM 5.6: *Let K be a PAC field of characteristic $\neq p$. Let $\mathcal{G}$ be a finitely generated cyclotomic pro-p pair. The following conditions are equivalent:*

(a) *there exists an extension F of K of relative transcendence degree 1 such that G_F is a pro-p group and $\mathcal{G} \cong \mathcal{G}(F)$;*
(b) *there exists an extension F of K of relative transcendence degree 1 and containing μ_p such that $\mathcal{G} \cong \mathcal{G}(F)$;*

(c) *there exist* $m_1, \ldots, m_n \in \{0,1\}$ *and separable algebraic extensions* $L_1, \ldots, L_n$ *of* K *containing* μ_p *such that*

$$\mathcal{G} \cong (\mathbb{Z}_p^{m_1} \rtimes \mathcal{G}(L_1)) * \cdots * (\mathbb{Z}_p^{m_n} \rtimes \mathcal{G}(L_n)).$$

Proof: (a)$\Rightarrow$(b): One should only observe that if G_F is pro-p then $\mu_p \subseteq F$.
(b)$\Rightarrow$(c): We use the method of [E4]. Since $K(\mu_p)$ is also PAC [FJ, Cor. 10.7], we may assume without loss of generality that $\mu_p \subseteq K$.

Since $\mathcal{G}$ is finitely generated so is $G_F(p)$. [Se, I-38, Cor.] therefore shows that $H^1(F)$ is finite. For $v \in \mathbb{P}(F/K)$ let $\hat{F}_v = F(p) \cap F_v^h$ and let $\hat{v}$ be the restriction of v^h to $\hat{F}_v$. Note that $\hat{v}$ is p-henselian, and $\hat{F}_v(p) = F(p)$. Let S_0 be the set of all $v \in \mathbb{P}(F/K)$ for which $H^2(\hat{F}_v) \neq 0$. For $v \in S_0$ one therefore has $\mathrm{cd}(G_{\hat{F}_v}(p)) \geq 2$, so $H^1(\hat{F}_v) \neq 0$. As $H^1(F)$ is finite, Lemma 5.3 then shows that S_0 is finite — say $S_0 = \{v_1, \ldots, v_s\}$ — and furthermore, $\mathrm{Res}\colon H^1(F) \to \prod_{i=1}^s H^1(\hat{F}_{v_i})$ is surjective. By Corollary 3.6, the map $\mathrm{Res}\colon H^2(F) \to \prod_{i=1}^s H^2(\hat{F}_{v_i})$ is injective. Lemma 5.2 now implies that $G_F(p) = G_{\hat{F}_{v_1}}(p) *_p \cdots *_p G_{\hat{F}_{v_s}}(p) *_p H$ for some finitely generated free pro-p group H. Write $H = H_1 *_p \cdots *_p H_t$ with $H_1, \ldots, H_t \leq H$ and $H_1 \cong \cdots \cong H_t \cong \mathbb{Z}_p$. Let $E_1, \ldots, E_t$ be the fixed fields of $H_1, \ldots, H_t$, respectively, in $F(p)$. By Remark 5.1(a),

$$\mathcal{G}(F) \cong \mathcal{G}(\hat{F}_{v_1}) * \cdots * \mathcal{G}(\hat{F}_{v_s}) * \mathcal{G}(E_1) * \cdots * \mathcal{G}(E_t).$$

It remains to analyze the structure of the factors in this free product. For each $1 \leq i \leq s$ let $m_i = \dim_{\mathbb{F}_p}(\Gamma_{v_i}/p)$. By Remark 5.1(c), $\mathcal{G}(\hat{F}_{v_i}) \cong \mathbb{Z}_p^{m_i} \rtimes \mathcal{G}(\bar{F}_{v_i})$. Since Γ_{v_i} is torsion-free and by [B, Ch. §10.3, Cor. 1],

$$m_i \leq \dim_{\mathbb{Q}}(\Gamma_{v_i} \otimes_{\mathbb{Z}} \mathbb{Q}) \leq \mathrm{tr.deg}(F/K) = 1.$$

Now $\bar{F}_{v_i}$ is an algebraic extension of K, and by Remark 5.1(b), $\mathcal{G}(\bar{F}_{v_i}) \cong \mathcal{G}(\bar{F}_{v_i} \cap K_{\mathrm{sep}})$. Hence the free factor $\mathcal{G}(\hat{F}_{v_i})$ has the desired structure.

Finally we consider the pairs $\mathcal{G}(E_j)$, $j = 1, \ldots, t$. If $\chi_{E_j,p} = 1$ then $\mathcal{G}(E_j) \cong \mathbb{Z}_p \rtimes (1,1) \cong \mathbb{Z}_p \rtimes \mathcal{G}(K(p))$. If $\chi_{E_j,p} \neq 1$ then $K(p) \not\subseteq E_j$. Therefore the image of the epimorphism $\mathrm{Res}\colon G_{E_j}(p) \to G_{K(p)\cap E_j}(p)$ is non-trivial. As $G_K(p)$ is torsion-free (Remark 5.1(d)) and $G_{E_j}(p) \cong \mathbb{Z}_p$, this epimorphism is necessarily bijective. Thus $\mathcal{G}(E_j) \cong \mathcal{G}(K(p) \cap E_j)$ has the desired structure in this case as well.

(c)$\Rightarrow$(a): Let $1 \leq i \leq n$. Since K is PAC, so is its algebraic extension L_i. By Remark 5.1(d), the natural epimorphism $G_{L_i} \to G_{L_i}(p)$ splits. Hence there exists a separable algebraic extension M_i of L_i such that G_{M_i} is pro-p and $\mathcal{G}(L_i) \cong \mathcal{G}(M_i)$ (Remark 5.1(b)).

Since a PAC field is infinite, we may choose distinct elements $\alpha_1,\dots,\alpha_n\in K$. Lemma 5.5 yields a henselian separable algebraic extension (F_i,v_i) of $M_i(t)=M_i(t-\alpha_i)$ such that v_i has rank 1, $m_i=\dim_{\mathbb{F}_p}(\Gamma_{v_i}/p)$, $(\overline{F_i})_{v_i}/M_i$ is purely inseparable, $v_i(t-\alpha_i)>0$, and G_{F_i} is pro-p, $i=1,\dots,n$. Then $v_1,\dots,v_n$ induce distinct valuations of rank 1 on $K(t)$. These valuations are independent, by [B, Ch. VI, §7.2, Th. 1] again. Let $F_i'=F_i\cap K(t)_{\rm sep}$, $i=1,\dots,n$, and let F_0 be a p-Sylow extension of $K(t)$ in $K(t)_{\rm sep}$. After replacing $F_1,\dots,F_n$ by $K(t)$-isomorphic copies we may assume that $F_1',\dots,F_n'$ extend F_0. Let F be their intersection. Proposition 5.4 implies that $G_F=G_{F_1'}*_p\cdots *_p G_{F_n'}$. By Remark 5.1, $\mathcal{G}(F)\cong\mathcal{G}(F_1')*\cdots*\mathcal{G}(F_n')$ and

$$\mathcal{G}(F_i')\cong\mathcal{G}(F_i)\cong\mathbb{Z}_p^{m_i}\rtimes\mathcal{G}((\bar{F}_i)_{v_i})\cong\mathbb{Z}_p^{m_i}\rtimes\mathcal{G}(M_i)\cong\mathbb{Z}_p^{m_i}\rtimes\mathcal{G}(L_i)$$

for $i=1,\dots,n$. Conclude that $\mathcal{G}(F)\cong\mathcal{G}$. ∎

We note that the pairs $\mathcal{G}(L_i)=(G_{L_i}(p),\chi_{L_i,p})$, $i=1,\dots,n$, in condition (c) of Theorem 5.6 have a very simple structure: by Remark 5.1(d) again, $G_{L_i}(p)$ is a finitely generated free pro-p group. By [E6, Lemma 1.1], pairs (H,θ), with H a finitely generated free pro-p group, are classified up to an isomorphism by the minimal number of generators of H and by $\mathrm{Im}(\theta)$. In particular, $\mathcal{G}(L_i)\cong(\mathbb{Z}_p,\theta_1)*\cdots*(\mathbb{Z}_p,\theta_m)$ for some continuous homomorphisms $\theta_1,\dots,\theta_m\colon\mathbb{Z}_p\to 1+p\mathbb{Z}_p$. In the terminology of [E5, §3] we therefore obtain:

COROLLARY 5.7: *Let K be a PAC field of characteristic $\neq p$ containing μ_p and let F be an extension of K of relative transcendence degree 1 such that $G_F(p)$ is finitely generated. Then $\mathcal{G}(F)$ has absolute elementary type.*

Now fix $p=2$. The elementary type conjecture in quadratic form theory predicts that the Witt rings of fields K with char $K\neq 2$ and $(K^\times:(K^\times)^2)<\infty$ can be constructed in finitely many steps from the Witt rings of finite fields, $\mathbb{C}$, $\mathbb{R}$, and the finite extensions of $\mathbb{Q}_2$, using the two fundamental constructions in the category of abstract Witt rings, called direct products and extensions (see [M, p. 123] for details). Jacob and Ware ([JW1], [JW2]) show that this happens if $\mathcal{G}(K)$ can be similarly constructed from the cyclotomic pro-2 pairs of such fields using free products and semi-direct products (i.e., $\mathcal{G}(K)$ has elementary type in the sense of [E5, §3]). Therefore Corollary 5.7 gives the following new evidence for the elementary type conjecture on Witt rings:

COROLLARY 5.8: *Suppose $p=2$ and let K, F be as in Corollary 5.7. Then the Witt ring $W(F)$ has elementary type.*

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